Realcompactness and spaces of vector-valued functions*

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Abstract

It is shown that the existence of a biseparating map between a large class of spaces of vector-valued continuous functions $A(X, E)$ and $A(Y, F)$ implies that some compactifications of X and Y are homeomorphic. In some cases, conditions are given to warrant the existence of a homeomorphism between the realcompactifications of X and Y; in particular we find remarkable differences with respect to the scalar context: namely, if E and F are infinite-dimensional and $T : C^*(X, E) \to C^*(Y, F)$ is a biseparating map, then the realcompactifications of X and Y are homeomorphic.

1 Introduction

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Given a completely regular space X, and a $\mathbb{K}$-normed space E, $C(X, E)$ and $C^*(X, E)$ denote the spaces of continuous functions and *bounded* continuous functions on X taking values on E, respectively. $C(X)$ and $C^*(X)$ will be the spaces $C(X, \mathbb{K})$ and $C^*(X, \mathbb{K})$, respectively.

Sometimes an algebraic relation between spaces of continuous functions $C(X)$ and $C(Y)$ may determine some kind of topological link between the spaces X and Y. For instance, it is well known that the existence of a ring isomorphism between the spaces $C(X)$ and $C(Y)$ produces a homeomorphism between the realcompactifications of X and Y (see [5, pp. 115-118] and [6]). Some kind of weakening on the conditions do not alter the result: if we replace "ring isomorphism" by "biseparating map" (see definition below), we keep the conclusion on the existence of a homeomorphism between the realcompactifications of X and Y ([1]). On the other hand, it is also true that if the realcompactifications υX of X and υY of Y are homeomorphic through a homeomorphism $h : \upsilon Y \to \upsilon X$, then there exists a biseparating map (in fact, a ring isomorphism) between

$C(X)$ and $C(Y)$, as the map sending each $f \in C(\upsilon X)$ into $f \circ h \in C(\upsilon Y)$ can be identified with an isomorphism between $C(X)$ and $C(Y)$.

Of course, if we study ring isomorphisms between spaces $C^*(X)$ and $C^*(Y)$, the conclusions we obtain are in general poorer, in the sense that we can no longer conclude the existence of a homeomorphism between the realcompact-ifications of X and Y, but just that the Stone-Čech compactifications of X and Y are homeomorphic. And this because every function $f \in C^*(X)$ can be extended to a continuous function in $C(\beta X)$. This implies that every ring isomorphism between $C^*(X)$ and $C^*(Y)$ can be regarded as one between $C(\beta X)$ and $C(\beta Y)$. Similar arguments also apply to biseparating maps between spaces of bounded continuous functions: the characteristics of these spaces yield a poor link between X and Y. In particular, as an example, if we take a realcompact space X which is not compact and define $Y := \beta X$, then we have that the map $T : C^*(X) \to C^*(Y)$ sending each $f \in C^*(X)$ into its extension to βX is clearly a biseparating map. Nevertheless X and βX are not homeomorphic. This example illustrates also the fact that, in general, the existence of a bisep-arating map from $C^*(X)$ onto $C^*(Y)$ does not imply the existence of a similar map between $C(X)$ and $C(Y)$. Indeed, if such a map existed, we would con-clude in our example that X and βX, which are both realcompact, should be homeomorphic, against our assumptions.

And what about spaces of vector-valued functions? Of course, in this context ring isomorphism does not make sense, but a close concept as biseparating map can be introduced here.

In particular, a natural question arises: what happens if we consider bisep-arating maps $T : C^*(X, E) \to C^*(Y, F)$? Clearly if E and F are infinite-dimensional, we can no longer act as in the scalar-valued case, because since E and F are not locally compact, bounded continuous functions cannot be ex-tended to functions on βX or βY attaining values in E or F. On the other hand, notice that classical techniques involving the study of ideals are no longer useful because our spaces do not have a ring structure.

What we obtain in this paper is a different conclusion. In our new context we are in a position to assure that the existence of a biseparating map be-tween $C^*(X, E)$ and $C^*(Y, F)$ allows us to claim that not only the Stone-Čech compactifications of X and Y are homeomorphic, but also their realcompact-ifications, a result that, as we mentioned above, is far from being true in the scalar case. On the other hand, this result is the best one we can expect: that is, we cannot expect a homeomorphism between both spaces X and Y.

Of course, we do not restrict our study to these special spaces, but we also obtain similar results for a large class of spaces, many of them containing unbounded functions. For instance, if we deal with spaces which do not contain "many" functions, the conclusions we get apply not to the spaces X and Y, but to some compactifications of them. But even in these cases, the structure of X and Y provides sometimes a strong link between them. In particular, this happens when we study biseparating maps between spaces of uniformly

continuous bounded functions.

2 Definitions and notation

All over the paper X and Y will be completely regular topological spaces, and E and F will be $\mathbb{K}$-normed spaces. $C(Y, F)$ and $C^*(Y, F)$ are defined, with the natural modifications, in the same way as $C(X, E)$ and $C^*(X, E)$ were defined at the beginning of this section. The same comments apply to $C(Y)$ and $C^*(Y)$. Finally, if X and Y are also complete metric spaces, we introduce $C_u^*(X, E)$ and $C_u^*(X, E)$ as the spaces of uniformly continuous bounded functions defined on X and Y, respectively, and taking values in E and F, respectively. Also in this case $C_u^*(X) = C_u^*(X, \mathbb{K})$, and $C_u^*(Y) = C_u^*(Y, \mathbb{K})$.

Definition 2.1 Given $f \in C(X, E)$, we define the cozero set of f as

$$c(f) := \{x \in X : f(x) \neq 0\}.$$

For the following two definitions, we assume that A, B are subrings of $C(X)$ and $C(Y)$, respectively, and that $A(X, E) \subset C(X, E)$, $A(Y, F) \subset C(Y, F)$ are an A-module and a B-module, respectively.

Definition 2.2 A map $T : A(X, E) \to A(Y, F)$ is said to be *separating* if it is additive and $c(Tf) \cap c(Tg) = \emptyset$ whenever $f, g \in A(X, E)$ satisfy $c(f) \cap c(g) = \emptyset$. Besides T is said to be *biseparating* if it is bijective and both T and T^{-1} are separating.

Equivalently, we see that an additive map $T : A(X, E) \to A(Y, F)$ is separating if $\|(Tf)(y)\| \, \|(Tg)(y)\| = 0$ for all $y \in Y$ whenever $f, g \in A(X, E)$ satisfy $\|f(x)\| \, \|g(x)\| = 0$ for all $x \in X$. Notice then that in particular every ring isomorphism between $C(X)$ and $C(Y)$ is clearly a biseparating map. Linearity of maps will be assumed at no point of this paper.

Definition 2.3 Let $T : A(X, E) \to A(Y, F)$ be a map and suppose that γX is a compactification of X. A point $x \in \gamma X$ is said to be a support point of $y \in Y$ if, for every neighborhood U of x in γX, there exists $f \in A(X, E)$ satisfying $c(f) \subset U$ such that $(Tf)(y) \neq 0$.

For a continuous map $f : X \to \mathbb{K}$, $f^{\beta X} : \beta X \to \mathbb{K} \cup \{\infty\}$ stands for the continuous extension to βX (the Stone-Čech compactification of X) into $\mathbb{K} \cup \{\infty\}$. In particular, given a continuous map $f : X \to E$, $\|f\|^{\beta X}$ will be the continuous extension to βX of $\|\cdot\| \circ f : X \to \mathbb{K} \cup \{\infty\}$. In the same way, if γX is a compactification of X, and $f : X \to \mathbb{K}$ is a continuous function which can be continuously extended to a map from γX into $\mathbb{K} \cup \{\infty\}$, we will denote by $f^{\gamma X}$ this natural extension. Also, for $\mathbf{e} \in E$, $\widehat{\mathbf{e}}$ will stand for the function constantly

equal to **e**. Finally, given $C \subset X$ and $D \subset \gamma X$, $\operatorname{cl} C$ and $\operatorname{cl}_{\gamma X} D$ will be their closures in X and γX, respectively.

Assume that A is a subring of $C(X)$ which separates each point of X from each point of βX. In βX, we introduce the equivalence relation $\sim$, defined as $x \sim y$ whenever $f^{\beta X}(x) = f^{\beta X}(y)$ for every $f \in A$. In this way, we obtain the quotient space $\gamma X := \beta X / \sim$. It is easy to see that γX is a compactification of X, and that every $f \in A$ can be continuously extended to a map from γX into $\mathbb{K} \cup \{\infty\}$. On the other hand, this extension will be bounded if f is bounded.

Suppose that $A(X, E) \subset C(X, E)$ is an A-module, where A is a subring of X which separates each point of X from each point of βX. We say that $A(X, E)$ is *compatible* with A if, for every $x \in X$, there exists $f \in A(X, E)$ with $f(x) \neq 0$, and if, given any points $x, y \in \beta X$ with $x \sim y$, we have $\|f\|^{\beta X}(x) = \|f\|^{\beta X}(y)$ for every $f \in A(X, E)$. It is clear that, in this case, for each $f \in A(X, E)$, there exists a continuous extension $\|f\|^{\gamma X} : \gamma X \to \mathbb{K} \cup \{\infty\}$ of $\|\cdot\| \circ f$ to the whole space γX.

A subring $A \subset C(X)$ is said to be *strongly regular* if given $x_0 \in \gamma X$ and a nonempty closed subset K of γX which does not contain x_0, there exists $f \in A$ such that $f^{\gamma X} \equiv 1$ on a neighborhood of x_0 and $f^{\gamma X}(K) \equiv 0$.

Examples. 1. Suppose that X is a metric space. Then the spaces $C(X, E)$ and $C^*(X, E)$ are both $C^*(X)$-modules compatible with $C^*(X)$. Also, in this case, it is easy to see that $\gamma X = \beta X$.

2. Now suppose that X is a complete metric space. It is easy to see that $C_u^*(X, E)$ is a $C_u^*(X)$-module compatible with $C_u^*(X)$. In this case, in general, $\gamma X \neq \beta X$. On the other hand, it is immediate that we can embed isometrically our space $C_u^*(X)$ in $C(\gamma X)$. Also $C_u^*(X)$ is a closed subalgebra of $C(\gamma X)$ which separates points, it contains constants and, when $\mathbb{K} = \mathbb{C}$, it is closed under complex conjugation. Then, by the Stone-Weierstrass theorem, it coincides with $C(\gamma X)$. Consequently $C_u^*(X)$ is a strongly regular ring.

3. Suppose that Ω is a (not necessarily bounded) open subset of $\mathbb{R}^p$ ($p \in \mathbb{N}$). For a Banach space E, consider the set $C^n(\Omega, E)$ of all E-valued functions whose partial derivatives up to the order n exist and are continuous ($n \geq 1$). Take $A = C^n(\Omega, \mathbb{R})$, which is a strongly regular ring (see for instance [9, Corollary 1.2]). On the other hand, it is straightforward to see that $C^n(\Omega, E)$ is an A-module and that, in this case $\gamma \Omega = \beta \Omega$.

4. In a similar way, if Ω is a *bounded* open subset of $\mathbb{R}^p$ ($p \in \mathbb{N}$) and E is a Banach space, we define $C^n(\bar{\Omega}, E)$ as the subspace of $C^n(\Omega, E)$ of those functions whose partial derivatives up to order n admit continuous extension to the boundary of Ω. Clearly, we can view $C^n(\bar{\Omega}, E)$ as a subset of $C(\operatorname{cl}_{\mathbb{R}^p} \Omega, E)$ As above, $A = C^n(\bar{\Omega}, \mathbb{R})$ is a strongly regular ring for which $\gamma \Omega = \operatorname{cl}_{\mathbb{R}^p} \Omega$, and $C^n(\bar{\Omega}, E)$ is an A-module.

A subring $A \subset C(X)$ is said to be *normal* if given two disjoint closed subsets K, L of βX, there exists $f \in A$, $0 \leq f \leq 1$, satisfying $f^{\beta X}(K) \equiv 1$ and

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$f^{\beta X}(L) \equiv 0$. It is said to be *local* if $f \in C(X)$ belongs to A whenever for every $x \in X$ there exist an open neighborhood $U(x)$ of x and $f_x \in A$ such that $f \equiv f_x$ in $U(x)$.

It is clear that each normal subring A of $C(X)$ is also strongly regular, and that every A-module $A(X, E) \subset C(X, E)$ must be compatible with A.

In this paper we will assume we are in one of the following three situations.

- **Situation 1.** $A(X, E) \subset C(X, E)$ and $A(Y, F) \subset C(Y, F)$ are an A-module and a B-module compatible with A and B, respectively, where $A \subset C(X)$ and $B \subset C(Y)$ are strongly regular rings. Also, in the case when $\gamma X \neq \beta X$ and $\gamma Y \neq \beta Y$, we also assume that for every $x \in \beta X$ and $y \in \beta Y$, there exist $f \in A(X, E)$ and $g \in A(Y, F)$ satifying $\|f\|^{\beta X}(x) \neq 0$, $\|g\|^{\beta Y}(y) \neq 0$.

- **Situation 2.** $A(X, E) \subset C(X, E)$ is an A-module and $A(Y, F) \subset C(Y, F)$ is a B-module, where $A \subset C(X)$ and $B \subset C(Y)$ are normal local rings.

- **Situation 3.** E and F are infinite-dimensional, and $A(X, E) = C^*(X, E)$, $A(Y, F) = C^*(Y, F)$.

Notice that when we are in Situations 2 or 3, then we are also in Situation 1. Also it is clear that when we are in Situations 2 and 3, then $\gamma X = \beta X$ and $\gamma Y = \beta Y$.

We will denote by υX and υY the realcompactifications of X and Y, respectively.

All over the paper the word "homeomorphism" will be synonymous with "surjective homeomorphism".

3 Main results

We first state a general result. Even if we may not have many functions in our spaces we can link the structures of some compactifications of X and Y.

Theorem 3.1 *Suppose that we are in Situation 1. If $T : A(X, E) \to A(Y, F)$ is a biseparating map, then γX and γY are homeomorphic.*

In some contexts, such as when X and Y have special structures, we can even ensure the existence of a homeomorphism between both spaces.

Corollary 3.2 *Assume that we are in Situation 1. Suppose that, for $A(X, E)$ and $A(Y, F)$, we have $\gamma X = \beta X$, and $\gamma Y = \beta Y$, respectively. If X and Y are first-countable spaces and $T : A(X, E) \to A(Y, F)$ is biseparating, then X and Y are homeomorphic. In particular, if X and Y are open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, then $p = q$.*

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Corollary 3.3 *Let $A(X, E) = C_u^*(X, E)$, $A(Y, F) = C_u^*(Y, F)$. If $T : A(X, E) \to A(Y, F)$ is a biseparating map, then X and Y are uniformly homeomorphic, that is, there exists a homeomorphism $h : Y \to X$ such that both h and h^{-1} are uniform maps.*

Notice that Corollary 3.2 applies to Examples 1, 3 and 4. In a similar way, Corollary 3.3 applies to Example 2.

Finally we state the main result of the paper, which applies to a large family of spaces of vector-valued functions.

Theorem 3.4 *Suppose that we are in Situations 2 or 3. If $T : A(X, E) \to A(Y, F)$ is biseparating, then υX and υY are homeomorphic.*

In general we cannot conclude, in Theorem 3.4, a statement like "X and Y are homeomorphic", as the following example shows.

Example. Take any space X which is not realcompact (for instance $W(\omega_1) := \{\sigma : \sigma < \omega_1\}$, where ω_1 denotes the first uncountable ordinal; see [5, 5.12]), and a realcompact normed space E. Recall that every normed space (or, more generally, every metrizable space) of nonmeasurable cardinal is realcompact ([5, p. 232]), and the assumption that all cardinal numbers are nonmeasurable is consistent with the axioms of set theory ([3, p. 217]). Next, each continuous map $f : X \to E$ can be extended to a continuous map $f^{\upsilon X} : \upsilon X \to E$. Clearly the map sending each $f \in C(X, E)$ into $f^{\upsilon X} \in C(\upsilon X, E)$ is biseparating but X and υX are not homeomorphic.

4 Proofs I

In this section we assume that we are in Situation 1.

Lemma 4.1 *For any $x \in X$, if U is an open neighborhood of x in γX, then there exists $f \in A(X, E)$ such that $x \in c(f)$ and $c(f) \subset U$.*

Proof. Take $g \in A(X, E)$ such that $x \in c(g)$, and $k \in A$ such that $k(x) = 1$ and $k^{\gamma X} \equiv 0$ outside U. It is easy to see that $f := gk \in A(X, E)$ does the job. $\square$

Lemma 4.2 *Let $T : A(X, E) \to A(Y, F)$ be a biseparating map. For $f, g \in A(X, E)$, if $c(f) \subset c(g)$, then $c(Tf) \subset \mathrm{cl}_{\gamma Y} c(Tg)$.*

Proof. Suppose that we can find $f, g \in A(X, E)$ with $c(f) \subset c(g)$ and $c(Tf) \not\subset \mathrm{cl}_{\gamma Y} c(Tg)$. Take $y \in c(Tf)$, $y \notin \mathrm{cl}_{\gamma Y} c(Tg)$. By Lemma 4.1 there exists $k \in A(Y, F)$ such that $k(y) \neq 0$ and $c(k) \cap \mathrm{cl}_{\gamma Y} c(Tg) = \emptyset$. Consequently $c(k) \cap c(Tg) = \emptyset$ and, since T^{-1} is separating, we deduce that $c(T^{-1}k) \cap c(g) = \emptyset$. Using the fact that T is separating we deduce that $c(k) \cap c(Tf) = \emptyset$. Since this is a contradiction, we conclude that $c(f) \subset \mathrm{cl}_{\gamma Y} c(Tg)$. $\square$

Lemma 4.3 *Let $T : A(X, E) \to A(Y, F)$ be a biseparating map. Then for each $y \in Y$, there exists a unique support point of y in γX.*

Proof. Take any $y \in Y$ and define $I_y := \{f \in A(Y, F) : y \in c(f)\}$. Now consider $H(y) := \bigcap_{f \in I_y} \mathrm{cl}_{\gamma X} \, c(T^{-1} f)$.

Claim 1. *$H(y)$ is nonempty.*

We are going to see that the family $\{\mathrm{cl}_{\gamma X} \, c(T^{-1} f) : f \in I_y\}$ satisfies the finite intersection property. Take $f_1, f_2, \ldots f_n \in I_y$. By Lemma 4.1, we can consider $f \in I_y$ such that $c(f) \subset \bigcap_{i=1}^n c(\|f_i\|^{\gamma Y})$. This gives us $c(f) \subset \bigcap_{i=1}^n c(f_i)$. Then, by Lemma 4.2 applied to T^{-1}, we have that $c(T^{-1} f) \subset \mathrm{cl}_{\gamma X} \, c(T^{-1} f_i)$ for every $i \in \{1, 2, \ldots, n\}$, which implies that $\bigcap_{i=1}^n \mathrm{cl}_{\gamma X} \, c(T f_i) \neq \emptyset$. By the compactness of γX, we have that $H(y)$ is nonempty, and the claim is proved.

Claim 2. *For each $y_0 \in Y$, $H(y_0)$ consists of just one point.*

Assume on the contrary that there exist two different points $x_1, x_2 \in H(y_0)$. Take $f_0 \in A(Y, F)$ such that $f_0(y_0) \neq 0$. By the definition of $H(y_0)$, we have that both x_1 and x_2 belong to $\mathrm{cl}_{\gamma X} \, c(T^{-1} f_0)$. Now consider a closed neighborhood U_2 of x_2 such that $x_1 \notin U_2$. Next, since A is strongly regular, we can take $g_1 \in A$ such that $g_1^{\gamma X} \equiv 1$ on a neighborhood U_1 of x_1, and $g_1^{\gamma X} \equiv 0$ on U_2. It is clear that

$$f_0 = T(g_1(T^{-1} f_0)) + T((1 - g_1)(T^{-1} f_0)),$$

and consequently y_0 belongs to

$$c(T(g_1(T^{-1} f_0)))$$

or to

$$c(T((1 - g_1)(T^{-1} f_0))).$$

We assume without loss of generality that $y_0 \in c(T(g_1(T^{-1} f_0)))$.

Now, since $y_0 \in c(T(g_1(T^{-1} f_0)))$ and $x_2 \in H(y_0)$, then x_2 belongs to $\mathrm{cl}_{\gamma X} \, c(g_1(T^{-1} f_0))$, which is not true by construction.

We conclude that $H(y_0)$ contains just one point.

Claim 3. *Given $y_0 \in Y$, if $x \in H(y_0)$, then x is a support point of y_0.*

Consider an open neighborhood U of $x \in H(y_0)$ in γX. We have to prove that there exists $g_0 \in A(X, E)$ such that $c(g_0) \subset U$ and $(T g_0)(y_0) \neq 0$.

Take $f_0 \in A(Y, F)$ such that $f_0(y_0) \neq 0$. Of course, if $c(T^{-1} f_0)$ is contained in U, we get the result by defining $g_0 := T^{-1} f_0$, so we suppose this is not the case.

Take $g_1 \in A$ such that $\mathrm{cl}_{\gamma X} \, c(g_1) \subset U$ and $g_1^{\gamma X} \equiv 1$ on a neighborhood of x. It is clear that

$$T^{-1} f_0 = g_1 T^{-1} f_0 + (1 - g_1) T^{-1} f_0,$$

and, as above, as a consequence we have that

$$(T(g_1 T^{-1} f_0))(y_0) \neq 0$$

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or
$$(T((1 - g_1)T^{-1}f_0))(y_0) \neq 0.$$

But notice that if the latter holds, since $x \in H(y_0)$ we should have $x \in \mathrm{cl}_{\gamma X}\, c((1 - g_1)T^{-1}f_0)$, which is not the case. Consequently, defining $g_0 := g_1 T^{-1} f_0$ we are done. $\square$

The previous lemma allows us to define a map $h : Y \to \gamma X$, sending each point $y \in Y$ into its support point $h(y) \in \gamma X$.

Lemma 4.4 *Let $T : A(X, E) \to A(Y, F)$ be a biseparating map. Suppose that $h(y) = x$ for some $y \in Y$, and that $f \in A(X, E)$ satisfies $\|f\|^{\gamma X} \equiv 0$ on a neighborhood of x. Then $Tf \equiv 0$ on a neighborhood of y.*

Proof. Take an open neighborhood U of x in γX such that $\|f\|^{\gamma X} \equiv 0$ in U. By the definition of support point, we can take $g \in A(X, E)$ such that $c(g) \subset U$ and $(Tg)(y) \neq 0$. Since T is biseparating, and $c(f) \cap c(g) = \emptyset$, we deduce that $Tf \equiv 0$ in $c(Tg)$, which is a neighborhood of y. $\square$

Now the following corollary follows easily.

Corollary 4.5 *Let $T : A(X, E) \to A(Y, F)$ be a biseparating map. Suppose that $y \in Y$, and that $f \in A(X, E)$ satisfies $(Tf)(y) \neq 0$. Then $h(y) \in \mathrm{cl}_{\gamma X}\, c(f)$.*

Lemma 4.6 *Given a biseparating map $T : A(X, E) \to A(Y, F)$, the associated map $h : Y \to \gamma X$ is continuous, and its range is dense in γX.*

Proof. We shall see first that h is continuous at every point of Y. Take $y_0 \in Y$ and an open neighborhood U of $h(y_0)$ in γX. By the definition of support point, there exists $g \in A(X, E)$ such that $\mathrm{cl}_{\gamma X}\, c(g) \subset U$ and $(Tg)(y_0) \neq 0$. Thus $c(Tg)$ is an open neighborhood of y_0. By Corollary 4.5, $h(c(Tg)) \subset U$ and we are done.

On the other hand, taking into account that T is injective, a similar reasoning yields that for every open subset U of γX, there are points $h(y)$ in U, and consequently the range of h is dense in γX. $\square$

Lemma 4.7 *The map h can be extended to a continuous map from γY onto γX.*

Proof. Obviously, as a consequence of the previous lemma, we can extend h to a continuous map $\widehat{h}$ from βY onto γX, so the result is true if $\gamma Y = \beta Y$.

Thus we assume that $\gamma Y \neq \beta Y$.

Claim 1. *Let $y_0 \in \beta Y$. Given an open neighborhood U of $\widehat{h}(y_0)$ in γX, there exists $f \in A(X, E)$ such that $c(f) \subset U$ and $\|Tf\|^{\beta Y}(y_0) \neq 0$.*

Notice that, by hypothesis, since $\gamma Y \neq \beta Y$, we have that there exists $g \in A(Y, F)$ such that $\|g\|^{\beta Y}(y_0) \neq 0$. Now take $k \in A$ such that $k^{\gamma Y} \equiv 0$ outside U and $k^{\gamma Y} \equiv 1$ on an open neighborhood, say V, of $\widehat{h}(y_0)$. It is clear that, if we

define $f := kT^{-1}g$, then $c(f) \subset U$. Also, $\widehat{h}^{-1}(V)$ is a neighborhood of y_0. Now, if we take $y \in \widehat{h}^{-1}(V) \cap Y$, then $h(y) \in V$, and consequently $\left\| f - T^{-1}g \right\|^{\gamma X} \equiv 0$ on a neighborhood of $h(y)$. By Lemma 4.4, $(Tf)(y) = g(y)$. Since this holds for every $y \in \widehat{h}^{-1}(V) \cap Y$, we conclude that $\|Tf\|^{\beta Y}(y_0) = \|g\|^{\beta Y}(y_0) \neq 0$.

Claim 2. *Given $y_1, y_2 \in \beta Y$, if $y_1 \sim y_2$, then $\widehat{h}(y_1) = \widehat{h}(y_2)$.*

Suppose that $y_1 \sim y_2$, $y_1 \neq y_2$, and that $\widehat{h}(y_1) \neq \widehat{h}(y_2)$. Take disjoint open subsets U and V of $\widehat{h}(y_1)$ and $\widehat{h}(y_2)$, respectively, in γX. Now, by Claim 1, there exist $f_1, f_2 \in A(X, E)$ such that $c(f_1) \subset U$, $c(f_2) \subset V$, and $\|Tf_1\|^{\beta Y}(y_1) \neq 0$, $\|Tf_2\|^{\beta Y}(y_2) \neq 0$. But, since T is separating and $c(f_1) \cap c(f_2) = \emptyset$, we deduce that $\|(Tf_1)(y)\| \, \|(Tf_2)(y)\| = 0$ for every $y \in Y$. Clearly this must force to $\|Tf_1\|^{\beta Y}(y_2) = 0$. This contradicts the fact that $A(Y, F)$ is compatible with B, which means in particular that $\|g\|^{\beta Y}(y_1) = \|g\|^{\beta Y}(y_2)$ for every $g \in A(Y, F)$.

Finally, because of Claim 2, given $y \in \gamma Y$ we can define the image of y as the image by $\widehat{h}$ of any of the elements of its equivalence class. It is clear that this determines a surjective continuous map from γY onto γX which is an extension of h. $\qquad\qquad\square$

The extension map given in Lemma 4.7 will also be called h.

Proof of Theorem 3.1. We will show that h is a homeomorphism. We are going to find an inverse for the map h. It is clear that since T is biseparating, we can construct a function $k : \gamma X \to \gamma Y$ associated to T^{-1}, which is an extension of a map from X into γY sending each point of X into its support point for T^{-1}. We just have to prove that k is the inverse map of h.

Claim 1. *If $y_0 \in Y$, then $k(h(y_0)) = y_0$.*

Assuming the contrary, suppose that $y_0 \in Y$ satisfies $y_0 \neq k(h(y_0))$. Then consider U, V two open neighborhoods in γY of y_0 and $k(h(y_0))$ respectively, such that $U \cap V = \emptyset$.

Next, applying Lemma 4.1, take $f \in A(Y, F)$ such that $c(f) \subset U$ and $f(y_0) \neq 0$. By Corollary 4.5, we have that $h(y_0)$ belongs to $\mathrm{cl}_{\gamma X}\, c(T^{-1}f)$.

On the other hand, as $k : \gamma X \to \gamma Y$ is continuous there exists an open neighborhood U_1 of $h(y_0)$ in γX such that if $x \in U_1$, then $k(x) \in V$. Also since $h(y_0) \in \mathrm{cl}_{\gamma X}\, c(T^{-1}f)$, then $U_1 \cap c(T^{-1}f) \neq \emptyset$, so we may take

$$x_0 \in U_1 \cap c(T^{-1}f).$$

Since $x_0 \in U_1$, $k(x_0) \in V$. But x_0 also belongs to X, and then, by the definition of support point, there exists $g \in A(Y, F)$ such that $c(g) \subset V$ and $(T^{-1}g)(x_0) \neq 0$.

So we have first that $c(f) \cap c(g) = \emptyset$. But it is clear that

$$x_0 \in c(T^{-1}f) \cap c(T^{-1}g),$$

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contradicting the fact that T^{-1} is separating.

As a consequence Claim 1 is proved.

Claim 2. *If $y_0 \in \gamma Y - Y$, then $k(h(y_0)) = y_0$.*

Take a net (y_α) in Y converging to $y_0 \in \gamma Y - Y$. Since both h and k are continuous, we have that the net $(k(h(y_\alpha)))$ converges to $k(h(y_0))$. But by Claim 1, $k(h(y_\alpha)) = y_\alpha$ for every α. This implies that it converges to y_0, and consequently $k(h(y_0)) = y_0$. So Claim 2 is proved.

As a consequence we easily conclude that k is the inverse map of h, and that both are homeomorphisms. □

Proof of Corollary 3.2. By the previous theorem, we have that βX and βY are homeomorphic. Also, since the only points of βX having a countable base of neighborhoods belong to X ([5, 9.7]), and the same applies to Y, we easily conclude that X and Y are homeomorphic. Finally, in the special case when X and Y are open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, this fact implies that $p = q$ (see for instance [4, p. 120]). □

Proof of Corollary 3.3. In [2, Lemma 3.4], it is proved, in a different context, that every point of X is a G_δ-set in γX, and that, on the contrary, no point in $\gamma X - X$ is a G_δ-set in γX. Consequently it follows that h is a homeomorphism from Y onto X.

Then we have a map $S : C(\gamma X) \to C(\gamma Y)$, defined as $(Sf)(y) = f(h(y))$ for every $f \in C(\gamma X)$ and every $y \in \gamma Y$. This map is easily seen to be bijective. Now, since $C_u^*(X)$ and $C_u^*(Y)$ can be identified with $C(\gamma X)$ and $C(\gamma Y)$, respectively (see Example 2), we may consider S as a map from $C_u^*(X)$ onto $C_u^*(Y)$, where it is also defined as $Sf = f \circ h$. As a consequence, for each $f \in C_u^*(X)$, $f \circ h$ belongs to $C_u^*(Y)$.

But, on the other hand, we can prove as in [7, Theorem 2.3] (see also the Remark after it) that if $f \circ h \in C_u^*(Y)$ whenever $f \in C_u^*(X)$, then h is uniformly continuous. Since the same process works also for h^{-1}, then the theorem is proved. □

5 Proofs II

In this section we will assume that we are in Situations 2 or 3.

Recall that in this context, if we look at the equivalence relation introduced in Section 2, then $\gamma X = \beta X$ and $\gamma Y = \beta Y$, so previous results can be applied here.

We start with a result whose proof is easy from Lemma 4.4 and the fact that h is a homeomorphism.

Lemma 5.1 *Let $T : A(X, E) \to A(Y, F)$ be a biseparating map. Suppose that $y \in \beta Y$, and that $f \in A(X, E)$ satisfies $\|f\|^{\beta X} \equiv 0$ on an open set U of βX. Then $\|Tf\|^{\beta Y} \equiv 0$ on the open set $h^{-1}(U) \subset \beta Y$.*

Proposition 5.2 *If $T : A(X, E) \to A(Y, F)$ is a biseparating map, then for every $y \in Y$, $h(y) \in \upsilon X$.*

Proof. Take $y_0 \in Y$, and suppose that $h(y_0) \in \beta X - \upsilon X$. Then there exists a sequence (U_n) of open neighborhoods of $h(y_0)$ in βX such that $\mathrm{cl}_{\beta X}\, U_{n+1} \subset U_n$, $\mathrm{cl}_{\beta X}\, U_{n+1} \neq U_n$, for every $n \in \mathbb{N}$ and $X \cap \bigcap_{n=1}^{\infty} U_n = \emptyset$ (see for instance [3, Theorem 3.11.10]). It is clear that

$$h(y_0) \in \mathrm{cl}_{\beta X} \bigcup_{n \in \mathbb{N}} (U_n - \mathrm{cl}_{\beta X}\, U_{n+2}).$$

Since h is a homeomorphism, we deduce that $y_0 \in \mathrm{cl}_{\beta Y} \bigcup_{n \in \mathbb{N}} h^{-1}(U_n - \mathrm{cl}_{\beta X}\, U_{n+2})$. Consequently, if we define

$$V_1 \;:=\; \bigcup_{n \in \mathbb{N}} h^{-1}(U_{4n} - \mathrm{cl}_{\beta X}\, U_{4n+2}),$$

$$V_2 \;:=\; \bigcup_{n \in \mathbb{N}} h^{-1}(U_{4n-1} - \mathrm{cl}_{\beta X}\, U_{4n+1}),$$

$$V_3 \;:=\; \bigcup_{n \in \mathbb{N}} h^{-1}(U_{4n-2} - \mathrm{cl}_{\beta X}\, U_{4n}),$$

and

$$V_4 := \bigcup_{n \in \mathbb{N}} h^{-1}(U_{4n-3} - \mathrm{cl}_{\beta X}\, U_{4n-1}),$$

then we have that y_0 belongs to one of the sets $\mathrm{cl}_{\beta Y}\, V_1, \mathrm{cl}_{\beta Y}\, V_2, \mathrm{cl}_{\beta Y}\, V_3$ or $\mathrm{cl}_{\beta Y}\, V_4$. We assume without loss of generality that $y_0 \in \mathrm{cl}_{\beta Y}\, V_1$.

At this point we split the proof into two cases.

- *Case 1. Assume we are in Situation 3.*

  It is clear that, if for each $n \in \mathbb{N}$, we define $W_n := h^{-1}(U_{4n} - \mathrm{cl}_{\beta X}\, U_{4n+2})$, then $\mathrm{cl}\, W_n \cap \mathrm{cl}\, W_m = \emptyset$, for $n \neq m$. Consider now a sequence (g_n) in B such that, for every $n \in \mathbb{N}$, $0 \leq g_n \leq 1$, $g_n \equiv 1$ on $Y \cap W_n$, and $g_n \equiv 0$ outside $Y \cap h^{-1}(U_{4n-1} - \mathrm{cl}_{\beta X}\, U_{4n+3})$.

  Applying Riesz's Lemma ([8, Theorem 1.3.2]), we can take a sequence $(\mathbf{e}_n)$ of norm one points in F such that $\|\mathbf{e}_n - \mathbf{e}_m\| \geq 1/2$ for $n \neq m$.

  It is also clear that, since T^{-1} is separating, then

  $$c(T^{-1}(g_n \mathbf{e}_n)) \cap c(T^{-1}(g_m \mathbf{e}_m)) = \emptyset$$

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if $n \neq m$. Define $f := \sum_{n=1}^{\infty} T^{-1}(g_n \mathbf{e}_n)$. Notice that, for each $x \in X$, $f(x)$ belongs to E.

Claim 1. *f is continuous.*

Notice that each function $\|g_n \mathbf{e}_n\|^{\beta Y} \equiv 0$ outside $\mathrm{cl}_{\beta Y} h^{-1}(U_{4n-1} - \mathrm{cl}_{\beta X} U_{4n+3})$, which implies by Lemma 5.1 applied to T^{-1} that

$$\left\|T^{-1}(g_n \mathbf{e}_n)\right\|^{\beta X} \equiv 0$$

outside $\mathrm{cl}_{\beta X}(U_{4n-1} - \mathrm{cl}_{\beta X} U_{4n+3})$. This is, we have that for every $n \in \mathbb{N}$, $c(T^{-1}(g_n \mathbf{e}_n)) \subset \mathrm{cl}_{\beta X}(U_{4n-1} - \mathrm{cl}_{\beta X} U_{4n+3})$. Now given any $x \in X$, there exists an open neighborhood U of x in X such that there are just a few numbers $k \in \mathbb{N}$ satisfying $U \cap \mathrm{cl}_{\beta X}(U_{4k-1} - \mathrm{cl}_{\beta X} U_{4k+3}) \neq \emptyset$, due to the construction of the sequence (U_n). This proves Claim 1.

Claim 2. *f belongs to $A(X, E)$.*

To prove it, we just need to show that f is bounded. Suppose on the contrary that the sequence $(\|T^{-1}(g_n \mathbf{e}_n)\|)$ is not bounded. For each $n \in \mathbb{N}$, set $a_n := \|T^{-1}(g_n \mathbf{e}_n)\|$. Since the sequence (a_n) is not bounded, we can extract a subsequence (which without loss of generality we shall assume it to be the whole (a_n)) with the property that $a_n \geq n^3$ for every $n \in \mathbb{N}$. Next consider the map

$$g := \sum_{n \in \mathbb{N}} \frac{g_n \mathbf{e}_n}{n^2}.$$

It is clear that, since each $\|g_n \mathbf{e}_n\| \leq 1$, then g belongs to $A(Y, F)$. Consequently $T^{-1}g$ exists and is a bounded function on X. We are going to see that this is not true, obtaining a contradiction. To this end take, for each $n \in \mathbb{N}$, $x_n \in X$ such that

$$\left\|(T^{-1}(g_n \mathbf{e}_n))(x_n)\right\| \geq \frac{a_n}{2} \geq \frac{n^3}{2}.$$

Now notice that, for each $n_0 \in \mathbb{N}$,

$$g = \frac{g_{n_0} \mathbf{e}_{n_0}}{n_0^2} + \sum_{n \neq n_0} \frac{g_n \mathbf{e}_n}{n^2},$$

where

$$c\left(\frac{g_{n_0} \mathbf{e}_{n_0}}{n_0^2}\right) \cap c\left(\sum_{n \neq n_0} \frac{g_n \mathbf{e}_n}{n^2}\right) = \emptyset.$$

Since T^{-1} is separating, we deduce that

$$c\left(T^{-1}\left(\frac{g_{n_0} \mathbf{e}_{n_0}}{n_0^2}\right)\right) \cap c\left(T^{-1}\sum_{n \neq n_0} \frac{g_n \mathbf{e}_n}{n^2}\right) = \emptyset,$$

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and this fact implies that

$$\left\|(T^{-1}g)(x_{n_0})\right\| = \frac{1}{n_0^2}\left\|(T^{-1}(g_{n_0}\mathbf{e}_{n_0}))(x_{n_0})\right\| \geq \frac{n_0}{2}.$$

Since this applies to every $n_0 \in \mathbb{N}$, we deduce that $T^{-1}g$ is not bounded. This contradiction allows us to conclude that f is bounded, and Claim 2 is proved.

Consequently we have that Tf belongs to $A(Y, F)$, and for each $n \in \mathbb{N}$,

$$\left\|f - T^{-1}(g_n\mathbf{e}_n)\right\|^{\beta X} \equiv 0$$

on $U_{4n} - \mathrm{cl}_{\beta X} U_{4n+2}$, which implies by Lemma 5.1 that

$$Tf \equiv g_n\mathbf{e}_n \equiv \mathbf{e}_n$$

in $Y \cap W_n$. Also, suppose that $(Tf)(y_0) = \mathbf{e}_0 \in F$. Then there exists a neighborhood U of y_0 in Y such that if $y, y' \in U$, then

$$\|(Tf)(y) - (Tf)(y')\| < \frac{1}{4}.$$

On the other hand, recall that $y_0 \in \mathrm{cl}_{\beta Y} V_1$ and, since $h(y_0) \notin \mathrm{cl}_{\beta X} (U_n - \mathrm{cl}_{\beta X} U_{n+2})$ for any $n \in \mathbb{N}$, then $y_0 \notin \mathrm{cl}_{\beta Y} W_n$ for any $n \in \mathbb{N}$. This implies that for every $k \in \mathbb{N}$,

$$y_0 \in \mathrm{cl}_{\beta Y} \bigcup_{n \geq k} W_n.$$

As a consequence, there are $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$, such that $U \cap W_{n_1} \neq \emptyset \neq U \cap W_{n_2}$, and then there are $y_{n_1} \in U \cap W_{n_1}$, $y_{n_2} \in U \cap W_{n_2}$. Also

$$\frac{1}{4} > \|(Tf)(y_{n_1}) - (Tf)(y_{n_2})\| = \|\mathbf{e}_{n_1} - \mathbf{e}_{n_2}\| \geq \frac{1}{2},$$

which is impossible.

- *Case 2. Assume we are in Situation 2.*

  In this case we follow a similar pattern of proof to the given above. Now consider a sequence (f_n) in A such that, for every $n \in \mathbb{N}$, $c(f_n) \subset X \cap U_{4n-2}$, and $f_n(x) = 1$ for every $x \in X \cap U_{4n}$. Define $g := \sum_{n=1}^{\infty} f_n$. Since A is local, it is easy to check that g belongs to A. Also it is easy to see that for every $n \in \mathbb{N}$, g is constantly equal to n on $X \cap (U_{4n} - \mathrm{cl}_{\beta X} U_{4n+2})$. Next take $f \in A(X, E)$ such that $(Tf)(y_0) = \mathbf{f}_0 \neq 0$. Suppose that $T(gf)(y_0) = \mathbf{f}_1 \in F$. Consider $n_0 \in \mathbb{N}$, $n_0 \|\mathbf{f}_0\|/2 > \|\mathbf{f}_1\| + 1$, and an open neighborhood $U(y_0)$ of y_0 in βY such that $h(U(y_0)) \subset U_{4n_0} \cap V$, where

  $$V := \left\{x \in \beta X : \|Tf\|^{\beta Y}(h^{-1}(x)) > \frac{\mathbf{f}_0}{2}\right\}.$$

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Since $h(y_0)$ belongs to $\mathrm{cl}_{\beta X} V_1$, then as above for every $k \in \mathbb{N}$,

$$y_0 \in \mathrm{cl}_{\beta Y} \bigcup_{n \geq k} W_n.$$

Now it is not difficult to see that there exists $k \in \mathbb{N}$, $k \geq n_0$, such that $h(U(y_0)) \cap (U_{4k} - \mathrm{cl}_{\beta X} U_{4k+2})$ is nonempty. Then if for $y_1 \in U(y_0) \cap Y$, $h(y_1)$ belongs to $U_{4k} - \mathrm{cl}_{\beta X} U_{4k+2}$, we have that $\|gf - kf\|^{\beta X}$ is constantly equal to zero in a neighborhood of $h(y_1)$. This means, by Lemma 5.1, that $T(gf - kf)(y_1) = 0$, and consequently,

$$\|T(gf)(y_1)\| = k \, \|(Tf)(y_1)\| \geq n_0 \frac{\|\mathbf{f_0}\|}{2} \geq \|\mathbf{f_1}\| + 1.$$

Since this happens for every open neighborhood of y_0, we deduce that $T(gf)$ is not continuous, which is not possible.

In both cases, we conclude that $h(y)$ belongs to υX for every $y \in Y$. □

Proof of Theorem 3.4. We have that the restriction of h to υY is continuous. Also, by Proposition 5.2, $h(y)$ belongs to υX for every $y \in Y$. Since υX is realcompact, we deduce that $h(\upsilon Y)$ is contained in υX. Since h^{-1}, for the same reason, maps elements of υX into elements of υY, and h^{-1} is continuous, we conclude that the restriction of h to υY is a homeomorphism onto υX. □

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